Goddard Problem with Bounded Thrust

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The problem considered is that of finding the thrust program, or control, as a piecewise continuous function of time, so that, with a fixed fuel supply, the summit altitude of a rocket in vertical flight is maximized. Under the assumptions that an optimal control exists and that the upper bound of thrust is at a sufficiently high magnitude, it is proved that the optimal control contains at most three subarcs. The analysis extends the previous results of Miele, Leitmann, and others to a larger class of drag functions. One of the drag functions, belonging to the class considered in the present analysis, is shown to correspond to a drag coefficient witnessing a sharp increase in the neighborhood of the speed of sound.

Introduction

GODDARD,[1] in 1919, posed the problem of delivering a rocket in vertical flight to a specified summit altitude with least fuel expenditure. In this paper, a version of the Goddard problem is considered: namely, with the thrust of the rocket engine limited by some maximum level, find the thrust program, or control, as a piecewise continuous function of time, so that with a fixed fuel supply the summit altitude is maximized. It is assumed for the analysis that such a control exists, and this control is defined as an optimal control. Miele[2] has established for the quadratic law of drag, and more generally for the power law of drag, that the optimal control contains at most three subarcs. Leitmann[3] derived the same result for the quadratic law of drag. The purpose of this paper is to extend these results of Miele, Leitmann, and others to a larger class of drag functions. The particular class of drag functions used in the present analysis contains as proper subsets the classes of drag functions considered by Miele[2] and Leitmann.[3] Moreover, the particular class of drag functions used in the analysis contains as a proper subset those drag functions that are a convex function of velocity. The main result of the paper is that, with thrust at a sufficiently high, but finite level, the optimal control contains at most three subarcs.

The Pontryagin maximum principle[4] was adopted for the analysis, since the control belongs to a closed set. More recently, one of several results given in a paper by Garfinkel[5] is a unique solution to the Goddard problem for drag as a convex function of velocity.† In Ref. 5, infinite thrust is admitted, and a unique solution is established by using sufficiency arguments from the classical calculus of variations. The results of the present paper were anticipated in the earlier work of Faulkner.[6] The use of the "switching function," in deriving results for the Goddard problem, appeared in the work of Breakwell.[7]

Problem Formulation

The equations of motion for a rocket in vertical flight are assumed to be given by the first-order system

$$dh/dt = v \tag{1}$$

$$dv/dt = \{[T - D(h, v)]/m\} - g \tag{2}$$

$$dm/dt = -(T/c) \tag{3}$$

where h, v, m represent the altitude, velocity, and the mass of the rocket at any instant of time. The thrust of the engine at any time is given by T, and the drag is assumed to be a function of the two variables h, v. Also, it is assumed that the acceleration of gravity g and the exhaust velocity c are constant. At the initial time t_0, we have

$$h(t_0) = 0 \qquad v(t_0) = 0 \qquad m(t_0) = m_0 \tag{4}$$

and at the final time t_1,

$$v(t_1) = 0 \qquad m(t_1) = m_f \tag{5}$$

The problem of interest is to find the piecewise continuous thrust T, or control, belonging to the closed set

$$0 \leq T \leq T_{\max} \tag{6}$$

so that, at the final time, which is open,

$$h(t_1) = \max h(t)|_{t=t_1} \tag{7}$$

Pontryagin Maximum Principle

As in Ref. 4, we introduce the function

$$H = \psi_0 v + \psi_1\{[(T - D)/m] - g\} + \psi_2 [-(T/c)] \tag{8}$$

and the adjoint system

$$d\psi_0/dt = -(\partial H/\partial h) = (\psi_1/m)(\partial D/\partial h) \tag{9}$$

$$d\psi_1/dt = -(\partial H/\partial v) = -\psi_0 + (\psi_1/m)(\partial D/\partial v) \tag{10}$$

$$d\psi_2/dt = -(\partial H/\partial m) = (\psi_1/m^2)(T - D) \tag{11}$$

We next rewrite (8) as

$$H = TS + \psi_1[-(D/m) - g] + \psi_0 v \tag{12}$$

where S is defined as the switching function

$$S = (\psi_1/m) - (\psi_2/c) \tag{13}$$

From part 1° of the Pontryagin maximum principle,[4] one observes that the function H given in Eq. (12) is maximized with respect to the optimum control T. We first classify the optimum control depending on whether S given in Eq. (13) is positive, negative, or zero for some nonzero time interval as follows:

$S > 0$ $\qquad T \equiv T_{\max}$, full thrust subarc (14)

$S \equiv 0$ $\qquad 0 \leq T \leq T_{\max}$, variable thrust subarc‡ (15)

$S < 0$ $\qquad T \equiv 0$, coasting subarc (16)

To complete the classification, we must include the situation

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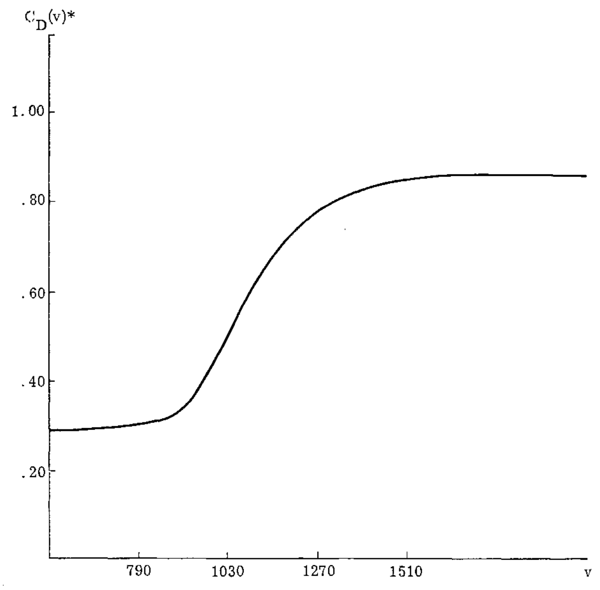
† This result has been extended in Ref. 8 to a wider class of drag functions.

‡ The variable thrust subarc is sometimes referred to as a singular subarc.



*CORRESPONDING $\lambda'(v)$ CURVE GIVEN IN FIG. 2

Fig. 1 Sounding rocket with hemispherical nose.

whereby S vanishes at some time t^*, and in addition where a positive δ does not exist, so that S vanishes for each t in $t^* - \delta \leq t \leq t^* + \delta$. Such a time t^* is defined as a switching point,§ and following the convention of Pontryagin et al.,[4] we assume that, at the time t^*, the control is given by the left-hand limit point

$$T(t^*) = T(t^* - 0) \tag{17}$$

From part 2^0 of the maximum principle,[4] one obtains from Eq. (12) for the optimal control T in $t_0 \leq t \leq t_1$

$$TS + \psi_1[-(D/m) - g] + \psi_0 v = 0 \tag{18}$$

Class D_1

We shall assume in this paper that the velocity v, except at the initial and final times, is positive along an optimal trajectory. From Eqs. (9–13) we derive the first derivative of the switching function yielding

$$dS/dt = (\psi_1 E/m^2 v) + (TS/mv) \tag{19}$$

where E is given by

$$E = v\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) - (D + mg) \tag{20}$$

We shall represent drag as [$k_1(v)$ is called the drag function]

$$D = w_0 k_1(v) e^{-\alpha h} \tag{21}$$

where w_0, α are positive constants, and $k_1(v)$ has continuous first and second derivatives with respect to v for $v > 0$. In addition $k_1(v)$ satisfies

$$k_1(0) = 0 \tag{22}$$

$$(d/dv)k_1(v) > 0,\ v > 0 \tag{23}$$

It follows from Eqs. (20) and (21) that

$$E = w_0 \lambda(v) e^{-\alpha h} - mg \tag{24}$$

where

$$\lambda(v) = v[(dk_1/dv) + (k_1/c)] - k_1 \tag{25}$$

We define the class D_1 to be those functions as represented in Eq. (21) for which $\lambda(v)$ as defined in Eq. (25) satisfies for $v > 0$

$$\lambda'(v) = v[(d^2k_1/dv^2) + (1/c)(dk_1/dv)] + (k_1/c) > 0 \tag{26}$$

Thus, if the drag is a convex function of velocity, it belongs to D_1. From Eqs. (22) and (23) it follows that $k_1(v)$ is positive for $v > 0$. It follows from Eq. (26) that $k_1(v)$ can possess a negative second derivative at some value of v and still give rise to a drag belonging to D_1. In this case, the drag is not a convex function of velocity (Figs. 1 and 2). We shall next establish 4 lemmas, only the first of which is a known result.

§ We omit those points $t = t^*$ for which there exists a $\delta > 0$ such that $S(t) > 0$ or $S(t) < 0$ on $t^* - \delta \leq t \leq t^* + \delta$ (except $t = t^*$).

Lemma 1

The optimal control begins with a full thrust subarc.

Proof: Since the rocket is initially at rest, it follows that the optimal control begins with a variable thrust or a full thrust subarc. Let us assume that the optimal control begins with a variable thrust subarc. It follows from Eqs. (15) and (19) that, along the variable thrust subarc [$\psi_1(t) > 0$, $t_0 \leq t < t_1$ (see Ref. 7)],

$$E = 0 \tag{27}$$

This well-known result appears in Refs. 2,3 and 5–7. Since from Eqs. (22, 24, and 25) the function E is initially negative, we conclude from Eq. (27) that the optimal control begins with a full thrust subarc.

Lemma 2

For drag belonging to D_1, a switching from a full thrust subarc to a coasting subarc is only possible at that altitude when fuel is exhausted.

Proof: At the switching point of a full-thrust subarc to a coasting subarc, the switching function S must vanish. Since the adjoint variable $\psi_1(t)$ is positive, except at the final time,[7] it follows from Eq. (19) that $E \leq 0$ at the switching point. Along the coasting subarc we get from (19)

$$dS/dt = \psi_1 E/m^2 v \tag{28}$$

From Eq. (24) it follows that along the coasting subarc where m is constant

$$dE/dt = w_0 e^{-\alpha h}[\lambda'(v)\dot{v} - \lambda(v)\alpha v] \tag{29}$$

Since $\dot{v}$ is negative along the coasting subarc, it follows from Eqs. (26) and (29) that, for drag belonging to D_1,

$$dE/dt < 0 \tag{30}$$

along the coasting subarc. Since in Ref. 7 the adjoint variable $\psi_1(t)$ is positive except at the final time, it follows from Eqs. (28) and (30) that, for drag belonging to D_1,

$$dS/dt < 0 \tag{31}$$

along the coasting subarc.

Let us assume that a switching is possible from a full thrust subarc to a coasting subarc at some time t^*. It follows from Eq. (16) that a positive δ exists, so that

$$S < 0 \qquad T \equiv 0 \text{ for all } t \text{ in } t^* < t \leq t^* + \delta \tag{32}$$

It follows from Eqs. (31) and (32) that the switching function that is negative at the time $t^* + \delta$ remains negative along a coasting subarc. Since the switching function will not assume the value of zero along the coasting subarc, it follows that the optimal control cannot switch from the coasting subarc. We conclude that, to satisfy the boundary condition given in Eq. (5), a switching from a full thrust subarc to a coasting subarc is only possible at that altitude when fuel is exhausted.

Lemma 3

For drag belonging to D_1, a switching from a variable thrust subarc to a coasting subarc is only possible at an altitude when fuel is exhausted.¶

Proof: The proof is identical to that proof offered for lemma 2, except that $S = 0$, $E \leq 0$ at the switching point is replaced by $S = 0$, $E = 0$, at the switching point.

Lemma 4

Suppose that $E > 0$ on a full thrust subarc. It follows that such a full thrust cannot follow a variable thrust subarc.

Proof: From Eq. (19) we have seen that, along a full thrust subarc,

$$dS/dt = (\psi_1 E/m^2 v) + (T_{\max} S/mv) \tag{33}$$

At the switching point of a variable thrust subarc to a full thrust subarc, both S and E vanish. Since E is positive along the full thrust subarc and $\psi_1(t)$ is positive,[7] it follows from Eq. (33) that along the full thrust subarc

$$dS/dt > 0 \tag{34}$$

Similar to the development of Eq. (32), we have that, if a switching from a variable thrust subarc to a full thrust subarc takes place at t^*, then positive δ exists, so that

$$S > 0 \qquad T \equiv T_{\max} \text{ for all } t \text{ in } t^* < t \leq t^* + \delta \tag{35}$$

It follows from Eqs. (34) and (35) that the switching function that is positive at the time $t^* + \delta$ remains positive along the full thrust subarc. Since the switching function will not assume the value of zero along the assumed full thrust subarc, the optimal control cannot switch from the full thrust subarc. A contradiction arises, since in such a case the boundary condition given in Eq. (5) cannot be satisfied.

We next offer a theorem that in particular will extend the results of Miele and Leitmann to the class D_1. To establish the theorem we shall assume the existence of an optimal control.

Theorem

For drag belonging to D_1 the optimal control contains at most three subarcs, provided that the upper bound of thrust is at a sufficiently high magnitude.

Proof: From lemma 1, the optimal control begins with a full thrust subarc. Two possibilities are next considered, namely, the optimal control switches from the full thrust subarc to either a coasting or to a variable thrust subarc.** From lemma 2, we have seen that the first possibility gives rise to a two subarc optimal control. From lemmas 3 and 4, the second possibility can give rise to a three subarc optimal control, provided that the requirement is met in lemma 4 that $E > 0$ along a full thrust subarc. We will next establish this result for drag belonging to D_1, by choosing the upper bound of thrust at a sufficiently high magnitude. First we will find an upper bound for velocities along an optimal trajectory. From Eqs. (1) and (2),

$$\dot{v} < -c\dot{m}/m \tag{36}$$

Integrating Eq. (36) yields

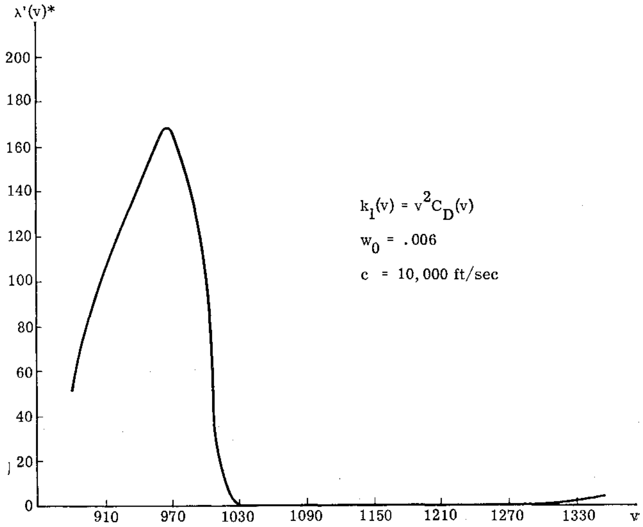
$$v < c \ln m_0/m \tag{37}$$

Using the boundary condition in Eq. (5), we establish an upper bound $v_{\max}$ for velocities along an optimal trajectory, namely,

$$v_{\max} = c \ln m_0/m_f \tag{38}$$

¶ This lemma offers some criteria for "getting off the singular subarc." From the preceding lemmas, one obtains that the optimal control cannot end with a full thrust subarc, since, in such a case, the boundary condition at $t = t_1$ cannot be satisfied. Also, since $E < 0$ at $t = t_1$, it follows that the optimal control cannot end with a variable thrust subarc. One concludes that the optimal control ends with a coasting subarc.

** Upper bound in Eq. (38) was established by G. Moyer of the Grumman Research Department.



* FOR THIS EXAMPLE $\lambda'(v) \geq 0$, $v > 0$. ALTHOUGH (26) IS MODIFIED, THE RESULTS OF THE ANALYSIS STILL GIVE THAT THE OPTIMAL CONTROL CONTAINS AT MOST THREE SUBARCS.

Fig. 2 Curve that defines a drag function.

We next rewrite Eq. (24) as

$$E = w_0 e^{-\alpha h \alpha h} [\lambda(v) - (mg/w_0)e^{\alpha h}] \tag{39}$$

It follows from Eq. (39) that the requirement in lemma 4, that $E > 0$ along a full thrust subarc, is met, provided that

$$\lambda'(v)\,\dot{v} + e^{\alpha h} gm/w_0 c[(T_{\max}/m) - \alpha cv] > 0 \tag{40}$$

However, Eq. (40) is satisfied for drag belonging to D_1 if we consider Eqs. (1–3, 21, 23, 26, and 38) and choose any upper bound of the thrust satisfying

$$T_{\max} \geq \max[\alpha c v_{\max} m_0,\ w_0 k_1(v_{\max}) + m_0 g] \tag{41}$$

We conclude that, for any drag belonging to D_1 and any upper bound of thrust satisfying Eq. (41), the optimal control is given by either two or three subarcs.

To complete the analysis, we derive from Eqs. (1–3 and 24) the optimum control along the variable thrust subarc yielding

$$T = \frac{\lambda'(v)(D + mg) + \alpha mv\lambda(v)}{\lambda'(v) + \lambda(v)/c} \tag{42}$$

We observe from Eqs. (26) and (42) that the optimal control does not vanish along the variable thrust subarc for drag belonging to D_1.

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